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Degenerate enveloping algebras of SU(3), SO(5), G_2 and SU(4)†

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Abstract. When they act on states of degenerate representations (for which one or more representation labels vanish) the generators of a simple group are no longer independent. Polynomial relations between them are conveniently interpreted as the vanishing of certain group tensors in the enveloping algebra. It turns out that the number of linearly independent λ-tensors in the degenerate enveloping algebra is equal to the number of $H\times U(1)\times\ldots\times U(1)$ scalars in the irreducible representation λ, where H is the subgroup whose Dynkin diagram is that corresponding to the vanishing labels and the U(1) subgroups correspond to the directions in weight space perpendicular to the hyperplane of H. Using generating function techniques we investigate the phenomenon for SU(3), SO(5), G_2 and SU(4). The consequences for subgroup labelling operators are discussed.

1. Introduction

In applications of group theory to physical problems one is often interested only in degenerate representations of the relevant group, i.e. representations for which one or more representation labels vanish. Examples are $(\lambda_1, 0)$ representations of SO(5) for nuclear quadrupole vibrations, $(\lambda_1, 0, 0)$ representations of SO(7) for octupole vibrations, $(\lambda_1, 0, \ldots, 0)$ representations of SU(n) for the n-dimensional isotropic oscillator. In this paper we investigate the enveloping algebra (polynomials in the generators) acting on (states of) degenerate representations of a simple group; the generators are then no longer independent, but satisfy polynomial relations, or syzygies, in addition to the commutation relations.

Generating function methods provide a convenient vehicle for discussing the enveloping algebra, whether acting on general or degenerate representations. Recently two of us (Couture and Sharp 1980; we refer to this paper hereafter as I) gave a generating function for irreducible tensors in the enveloping algebra of each simple group of rank three or less. The generating function is a rational function $\mathscr{G}$ of $l+1$ dummy variables $U, \Lambda_1, \ldots, \Lambda_l$ where l is the rank of the group. When expanded in a power series,

$$\mathscr{G} = \sum_u U^u \sum_\lambda c_{u\lambda} \Lambda^\lambda, \qquad \Lambda^\lambda \equiv \prod_{i=1}^{l} \Lambda_i^{\lambda_i}, \tag{1.1}$$

the generating function yields, as the coefficient $c_{u\lambda}$, the number of linearly independent

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tensors of degree u which transform by the representation $\lambda \equiv (\lambda_1, \ldots, \lambda_l)$. The Dynkin, or Cartan, label λ_i is defined by

$$\lambda_i = 2\langle M_\lambda \mid \alpha_i\rangle / \langle \alpha_i \mid \alpha_i\rangle, \tag{1.2}$$

where M_λ is the highest weight of the represntation λ, and α_i is the ith simple root. The generating function suggests a finite integrity basis, a finite set of 'elementary' tensors in terms of which all can be constructed as stretched products.

The syzygies which relate the generators acting on degenerate representations are best viewed as the vanishing of certain tensors in the enveloping algebra. The simpler form take by the enveloping algebra is conveniently described by presenting the degenerate form of its generating function.

In this paper we follow Kostant (1963) and Okubo (1975) in defining linear independence of tensors in terms of the linear independence of the matrices which represent them when they act on the (degenerate) representations under consideration; this is equivalent to the statement that we ignore group Casimir invariants.

In § 2 we describe how the degenerate functions are determined; the methods are necessarily more *ad hoc* than those available when the generators act on general representations. In § 3 we deal with SU(3) and SO(5); in § 4 we treat G_2 and SU(4). For each group considered we look at consequences for subgroup labelling operators (subgroup invariants other than Casimirs in the enveloping algebra of the group).

2. How the generating functions are determined

Straightforward (but tedious) algorithms exist for the determination of non-degenerate generating functions for a simple group G. An obvious procedure for determining degenerate generating functions is to construct the elementary tensors, and by allowing them to act on states transforming by the degenerate representations of interest, find the syzygies relating them. Much of this laborious task is avoided by exploiting a theorem of Kostant (1963).

Kostant showed that the number of independent λ tensors, other than scalars, in the enveloping algebra of a group G when it acts on the repsentation μ, is equal to the multiplicity of the representation μ in the Clebsch–Gordan series for $\lambda \otimes \mu$. According to a result of Weyl (1926) this multiplicity is the number of states of weight 0 in the representation λ, corrected for reflections in the hyperplanes $\lambda_i = -\mu_i - 1$. By choosing all the representation labels μ_i sufficiently large we see that, acting on general representations, there are as many independent λ tensors in the enveloping algebra as there are states of weight 0 in the representation λ.

Now let the λ tensors act on the degenerate representations μ for which a $(0 < a < l)$ of the representation labels μ_i, say $\mu_{i_1}, \ldots, \mu_{i_a}$, vanish. We may choose the non-vanishing μ_i large compared with the labels of λ. Then we must consider only Weyl reflections in the a hyperplanes $\lambda_{i_1} = -1, \ldots, \lambda_{i_a} = -1$, and other 'reflections' generated by them; these Weyl reflections are those of the rank-a subgroup H whose Dynkin diagram is obtained from that of G by removing vertices (and lines attached to them) corresponding to the non-vanishing labels μ_i. The number of independent λ tensors is thus

$$\eta^{-R_H}\chi_\lambda(\eta)\sum_{S_H}(-1)^{S_H}\eta^{S_H R_H}\big|_{\eta^0}, \qquad \eta^x \equiv \prod_{i=1}^{l}\eta_i^{x_i}. \tag{2.1}$$

In (2.1), η_i are l dummy variables which carry components of weight as their exponents; R_H is half the sum of the positive roots of H, or, equivalently, the sum of its fundamental dominant weights; S_H are the Weyl reflections of H; $\chi_\lambda(\eta) = \Sigma_{x_\lambda} \eta^{x_\lambda}$ is the character and x_λ the weights of the representation λ; the subscript η^0 is an instruction to retain only the coefficient of η^0. But (2.1) is just the number of $H \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$ invariants in the representation λ, where the l-a groups U(1) refer to components of weight in the directions orthogonal to those of H. If a generating function for $G \supset H \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$ branching rules is known (if not it can be computed), its subgroup scalar part is a generating function for λ tensors in the enveloping algebra acting on the degenerate representations μ.

A shortcoming of the above procedure is that it provides no information about the degrees of the surviving tensors or about group invariants. It provides the desired generating function, i.e. the degenerate analogue of (1.1), with group invariants set equal to zero, and with U, the dummy variable carrying the degree as its exponent, set equal to unity.

Further information about the degenerate generating function is obtained by considering its implication for subgroup invariants, including Casimirs, in the enveloping algebra. Knowledge of group–subgroup branching rules for the degenerate group representations under consideration tells us which subgroup Casimirs are independent, and hence gives information about the generating function for subgroup scalars, including the degrees of some elements of the integrity basis. In each case only a few variants of the degenerate generating function for group tensors in the degenerate enveloping algebra are consistent with the form deduced from Kostant's theorem and with the fact that it must imply an integrity basis which is a subset of that for the non-degenerate case (or other degenerate cases with fewer vanishing labels). Each possible form is converted to the corresponding generating function for subgroup scalars with the help of the group–subgroup branching rules (for subgroup scalars) and compared with what is known about subgroup Casimirs. Different subgroups provide different restrictions on the degenerate generating function. Another useful fact is the known presence in the degenerate generating function of the denominator factor $1 - U\Lambda^{\lambda_a}$ where λ_a is the adjoint representation (by which the generators transform).

Sometimes the procedures of the preceding paragraph provide complete information about tensors in the degenerate enveloping algebra (except for group invariants). Any remaining ambiguities may be resolved by constructing relevant tensors, or selected parts of them, and applying them to appropriate degenerate states. Many examples are found in the two following sections.

3. SU(3) and SO(5)

3.1. SU(3); $(\mu_1, 0)$ and $(0, \mu_2)$

We denote a λ tensor of degree u in the enveloping algebra $(\mu; \lambda) \equiv (u; \lambda_1, \lambda_2)$. The non-degenerate generating function is given by (I, equation (3.6)). In the degenerate enveloping algebra (acting on $(\mu_1, 0)$ or $(0, \mu_2)$ representations) the number of independent λ tensors is equal to the number of $\mathrm{SU}(2) \times \mathrm{U}(1)$ scalars in the representation λ. The relevant generating function is obtained from that for $\mathrm{SU}(3) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ branching rules (I, equation (2.3)) by setting $N_1 = 0$ and retaining the part of degree zero in N_2; the result is $(1-\Lambda_1\Lambda_2)^{-1}$. Since it must contain the factor $(1-U\Lambda_1\Lambda_2)^{-1}$ (the adjoint representation is (1, 1)), we conclude that, SU(3) scalars aside, the desired degenerate generating function is just $(1-U\Lambda_1\Lambda_2)^{-1}$.

3.2. SO(5); $(\mu_1, 0)$

The non-degenerate generating function for SO(5) is given by (I, equation (3.8)). In the degenerate enveloping algebra, acting on $(\mu_1, 0)$, the number of λ tensors is equal to the number of SU(2)×U(1) scalars in the representatin λ, where the SU(2) root is the second simple root of SO(5), i.e. one of the longer roots. The generating function for SU(2)×U(1) scalars is obtained from that for SO(5) ⊃ SU(2)×SU(2) branching rules (I, before equation (2.7*a*)) by setting $N_1=0$ and retaining the even part in N_2 with $N_2=1$; the result is $[(1-\Lambda_1^2)(1-\Lambda_2)]^{-1}$. Since the degenerate generating function contains a factor $(1-U\Lambda_1^2)^{-1}$ ((2, 0) is the adjoint representation), and since (2, 01) and (2, 02) are the only elementary tensors with $\lambda_1=0$, we conclude that the degenerate generating function is

$$(1+U^2\Lambda_2)[(1-U\Lambda_1^2)(1-U^2\Lambda_2^2)]^{-1} \tag{3.1a}$$

or else

$$[(1-U\Lambda_1^2)(1-U^2\Lambda_2)]^{-1}. \tag{3.1b}$$

To select one of these we consider their implication for SU(2)×SU(2) scalars. The SO(5) ⊃ SU(2)×SU(2) branching rules for $(\mu_1, 0)$ representations are given by the generating function $[(1-\Lambda_1 S)(1-\Lambda_1 T)]^{-1}$ which implies $s+t=\lambda_1$ (s and t are twice the usual angular momentum label). The SU(2) Casimirs are linearly independent, but the square of either can be expressed as a polynomial in the other. The generating function for subgroup scalars in the enveloping algebra is thus

$$(1+U^2)(1-U^2)^{-1}. \tag{3.2}$$

The generating function for SO(5) ⊃ SU(2)×SU(2) scalars, obtained from the equation preceding (I, (2.7*a*)) by setting $N_1=N_2=0$, is $(1-\Lambda_2)^{-1}$, which means that (3.1*a*) or (3.1*b*) are converted to generating functions for subgroup scalars in the enveloping algebra by setting $\Lambda_1=0$, $\Lambda_2=1$. Equation (3.1*a*) is correct, since it agrees with (3.2). The same conclusion is reached by considering scalars of the maximal subgroup SU(2)× U(1) in the enveloping algebra. Their generating function is obtained from (3.1*a*) or (3.1*b*) by keeping the part even in Λ_1 and even in Λ_2 and setting $\Lambda_1=\Lambda_2=1$; the correct result $[(1-U)(1-U^2)]^{-1}$ is obtained only from (3.1*a*).

3.3. SO(5); $(0, \mu_2)$

The number of λ tensors in the degenerate enveloping algebra on $(0, \mu_2)$ representations is equal to the number of SU(2)×U(1) scalars in the representation λ; this time SU(2) corresponds to the first (shorter) simple root and SU(2)×U(1) is maximal in SO(5). The generating function for SU(2)×U(1) branching rules is (Sharp and Lam 1969, case 4)

$$[(1-\Lambda_1 SZ)(1-\Lambda_1 SZ^{-1})(1-\Lambda_2 Z^2)(1-\Lambda_2 Z^{-2})]^{-1}[(1-\Lambda_1^2)^{-1}+\Lambda_2 S^2(1-\Lambda_2 S^2)^{-1}]. \tag{3.3}$$

The subgroup scalar part of this is obtained by setting $S=0$ and retaining the coefficient of Z^0; the result is $[(1-\Lambda_1^2)(1-\Lambda_2^2)]^{-1}$. Comparison with the non-degenerate generating function suggests that this can only imply

$$[(1-U\Lambda_1^2)(1-U^2\Lambda_2^2)]^{-1} \tag{3.4}$$

for the degenerate generating function for tensors in the enveloping algebra. This implies $(1-U^2)^{-1}$ and $[(1-U)(1-U^2)]^{-1}$ respectively for SU(2) × SU(2) and SU(2) × U(1) scalars in the enveloping algebra, in agreement with what we infer from the degenerate branching rules

3.4. *SO(5) ⊃ SU(2) labelling operators*

A generating function for missing label operators for SO(5) ⊃ SU(2) is given by Gaskell *et al* (1978, equation (31)). It defines the operators by giving their degrees in the $l=1$ tensor and in the $l=3$ tensor into which the SO(5) generators decompose. There are two missing labels and hence four algebraically independent missing label operators (Peccia and Sharp 1976).

The generating function for SO(5) ⊃ SU(2) branching rules is given by Gaskel *et al* (1978, equation (23)). Setting $A=0$ we obtain its SU(2) scalar part

$$(1+\Lambda_1^6\Lambda_2^3)[(1-\Lambda_1^4)(1-\Lambda_2^3)(1-\Lambda_1^4\Lambda_2^2)]^{-1}. \tag{3.5}$$

To get the generating function for subgroup scalars in the enveloping algebra acting on $(\mu_1, 0)$ representations, we multiply (3.5) by (3.1a) in which Λ_1 and Λ_2 have been replaced by Λ_1^{-1} and Λ_2^{-1} respectively and retain the coefficient of $\Lambda_1^0\Lambda_2^0$. Omitting a factor $(1-U^2)^{-1}$ corresponding to the SU(2) second degree Casimir, we obtain

$$(1+U^4+U^7+U^9)[(1-U^4)(1-U^6]^{-1} \tag{3.6}$$

as the generating function for missing label operators. We see that there are two functionally independent operators of degrees 4 and 6, as well as three more, of degrees 4, 7, 9, whose squares can be expressed as polynomials in the others (and in Casimirs). To identify these degenerate operators with those in the non-degenerate enveloping algebra (Gaskell *et al* 1978, equation (31)), we note that there are three degree 4 operators (1, 3), (2, 2), (3, 1), four degree 6 operators (2, 4), (3, 3), (5, 1), (6, 0), three of degree 7, (3, 4), (4, 3), (5, 2), and seven of degree 9, (3, 6), (4, 5), (5, 4), $(6, 3)_1$, $(6,3)_2$, (7,2), (8,1). The notation is (t, l), where t and l are the degrees in the $l=3$ and $l=1$ tensors respectively in the adjoint representation. In the degenerate case it is no longer meaningful to give the separate degrees t, l. The seven degree-9 operators become indistinguishable, as do the three of degree 7 and the four of degree 6. There is one linear relation connecting the three degree-4 operators, and the square of any one can be expressed as a polynomial in the others.

To find the generating function for subgroup scalars acting on $(0, \mu_2)$, we multiply (3.5) by (3.4) with the replacement $\Lambda_1 \to \Lambda_1^{-1}$, $\Lambda_2 \to \Lambda_2^{-1}$ and retain the part of degree 0 in Λ_1, Λ_2. The result is, omitting $(1-U^2)^{-1}$,

$$(1+U^9)[(1-U^4)(1-U^6)]^{-1}. \tag{3.7}$$

There is no labelling operator of degree 7 and the three of degree 4 are now indistinguishable. The operators defined by (3.7) are of interest in the problem of nuclear quadrupole vibrations.

4. G_2 and SU(4)

4.1. G_2; $(\mu_1, 0)$

The generating function for tensors in the non-degenerate enveloping algebra of G_2 is given by I, equation (4.9). Our notation is such that (1,0) is the septet, (0, 1) the fourteen-plet representation.

The number of independent λ tensors in the degenerate enveloping algebra on $(\mu_1, 0)$ representations is equal to the number of $SU(2)\times U(1)$ scalars in the representation λ, where SU(2) refers to the second, i.e. longer, simple root. The generating function for $G_2 \supset SU(2)\times SU(2)$ branching rules is given by Gaskell and Sharp (1981, equation (3.1)); the desired function for $SU(2)\times U(1)$ scalars is obtained by setting $S=0$, retaining the even part in T and setting $T=1$. The result is

$$(1+\Lambda_1\Lambda_2)[(1-\Lambda_1)(1-\Lambda_1^2)(1-\Lambda_2)(1-\Lambda_2^2)]^{-1}. \qquad (4.1)$$

Comparing this with the non-degenerate generating function, we conclude that the degenerate one is either

$$(1+U^c\Lambda_1\Lambda_2)[(1-U^3\Lambda_1)(1-U^a\Lambda_1^2)(1-U\Lambda_2)(1-U^b\Lambda_2^2)]^{-1} \qquad (4.2a)$$

or

$$(1+U^c\Lambda_1\Lambda_2)(1+U^3\Lambda_1)[(1-U^2\Lambda_1^2)(1-U^4\Lambda_1^2)(1-U\Lambda_2)(1-U^b\Lambda_2^2)]^{-1} \qquad (4.2b)$$

where $a=2$ or 4, $b=4$ or 8, $c=5$ or 7; we used the fact that the adjoint representation is (0, 1).

The generating function for $G_2 \supset SU(3)$ branching rules is given by Gaskell and Sharp (1981, equation (2.3)). For $(\mu_1, 0)$ representations it is

$$[(1-\Lambda_1 P)(1-\Lambda_1 Q)(1-\Lambda_1)]^{-1} \qquad (4.3)$$

where P, Q carry the SU(3) representation labels. There are no missing labels, and the SU(3) representation labels, or Casimirs, are independent. The generating function for SU(3) scalars in the degenerate G_2 enveloping algebra is therefore

$$[(1-U^2)(1-U^3)]^{-1}. \qquad (4.4)$$

The generating function for $G_2 \supset SU(3)$ scalars is $(1-\Lambda_1)^{-1}$. Therefore (4.2a) or (4.2b) is converted to the corresponding generating function for SU(3) scalars by setting $\Lambda_1=1$, $\Lambda_2=0$. Only (4.2a), with $a=2$, agrees with (4.4). The degrees b and c are not determined by considering subgroup Casimirs. The actual construction of the tensors is discussed in I. Replace the generators by states which transform by the adjoint representation; denote the highest component of the tensor as an unknown linear combination of those monomials in the states which have the necessary degree and weight. The coefficients are found by requiring that the generators corresponding to the simple roots annihilate the highest component. The states are then replaced by the corresponding generators and symmetrised as to order. Finally the highest component of the tensor may be applied to the lowest state of the degenerate representation (the work here is simplified by exploiting the fact that we need only to establish that the tensor in question does not vanish when applied to the degenerate representation; for example, of the 113 terms of (5, 11), only six need be computed, namely those which are products of generators none of which annihilate the lowest state). We find that (4, 02) and (5, 11) do not vanish on $(\mu, 0)$ representations. Hence we get finally the generating function for tensors in the degenerate enveloping algebra

$$(1+U^5\Lambda_1\Lambda_2)[(1-U^3\Lambda_1)(1-U^2\Lambda_1^2)(1-U\Lambda_2)(1-U^4\Lambda_2^2)]^{-1}. \qquad (4.5)$$

4.2. G_2; $(0, \mu_2)$

The number of independent λ tensors in the degenerate enveloping algebra on $(0, \mu_2)$ representations is equal to the number of SU(2) × U(1) scalars in the representation λ, where SU(2) refers to the first, i.e. shorter, simple root. The generating function for SU(2) × U(1) scalars is obtained from that for $G_2 \supset$ SU(2) × SU(2) branching rules (Gaskell and Sharp 1981, equation (3.1)) by setting $T=0$, keeping the even part in S and putting $S=1$. The result is

$$(1+\Lambda_1^3\Lambda_2)[(1-\Lambda_1^2)(1-\Lambda_1^3)(1-\Lambda_2)(1-\Lambda_2^2)]^{-1}$$

which implies, for the degenerate generating function,

$$(1+U^c\Lambda_1^3\Lambda_2)[(1-U^a\Lambda_1^2)(1-U^3\Lambda_1^3)(1-U\Lambda_2)(1-U^b\Lambda_2^2)^{-1}, \qquad (4.6)$$

where $a=2$ or 4, $b=4$ or 8, $c=5$ or 8. The SU(3) representation labels are independent in $(0, \mu_2)$ representations of G_2 so again we expect (4.4) as the generating function for SU(3) scalars in the enveloping algebra. But (4.6) is converted to a generating function for SU(3) scalars by setting $\Lambda_1=1$, $\Lambda_2=0$. Comparison with (4.4) shows $a=2$, and leaves b and c undetermined. Proceeding as in the $(\mu_1, 0)$ case we find $b=4$, $c=5$. The generating function finally is

$$(1+U^5\Lambda_1^3\Lambda_2)[(1-U^2\Lambda_1^2)(1-U^3\Lambda_1^3)(1-U\Lambda_2)(1-U^4\Lambda_2^2)]^{-1}. \qquad (4.7)$$

4.3. $G_2 \supset$ SU(2) × SU(2) labelling operators

There are no missing labels for degenerate representations in the chain $G_2 \supset$ SU(3).

For $G_2 \supset$ SU(2) × SU(2) there is one missing label for degenerate representations $(\mu_1, 0)$ or $(0, \mu_2)$. Setting $S=T=0$ in the generating function for $G_2 \supset$ SU(2) × SU(2) branching rules (Gaskell and Sharp 1981, equation (3.1)) gives $[(1-\Lambda_1^2)(1-\Lambda_2^2)]^{-1}$ as the generating function for SU(2) × SU(2) scalars in G_2 representations; hence we have to keep the even part in Λ_1 and Λ_2 of (4.5) or of (4.7) and set $\Lambda_1=\Lambda_2=1$ to get the respective generating functions for subgroup scalars in the degenerate enveloping algebra of the group. For $(\mu_1, 0)$, or for $(0, \mu_2)$, we get

$$(1+U^9)[(1-U^4)(1-U^6)]^{-1}; \qquad (4.8)$$

we have ignored a factor $(1-U^2)^{-2}$ corresponding to the Casimirs of SU(2) × SU(2). Equation (4.8) should be compared to the generating function for SU(2) × SU(2) scalars in the non-degenerate enveloping algebra of G_2 (Gaskell *et al* 1978, equation (30)). We can denote a scalar by (a, b, c) where a is its degree in the (3/2, 1/2) SU(2) × SU(2) tensor, b and c its degrees in the s and t SU(2) generators which comprise the generators of G_2. There are two scalars of degree 4, (2, 2, 0) and (2, 1, 1); there are four of degree 6, (4, 2, 0), (2, 3, 1), (4, 1, 1) and (4, 0, 2); there are five of degree 9, (6, 3, 0), (4, 4, 1), (6, 2, 1), (4, 3, 2), (6, 1, 2). According to (4.8) the two degree-4 scalars become indistinguishable, as do the four of degree 6, and the five of degree 9. We surmise that the degree-9 scalar is the commutator of those of degrees 4 and 6.

We turn to SO(3), the third maximal subgroup of G_2. A generating function for $G_2 \supset SO(3)$ branching rules is given by Gaskell and Sharp (1981, figure 1). With $L=0$ (and $A \to \Lambda_1$, $B \to \Lambda_2$) it becomes $M(\Lambda_1, \Lambda_2, 0)$, a generating function for SO(3) scalars in G_2 representations. To convert it to a generating function for SO(3) scalars in a degenerate G_2 enveloping algebra, multiply it by (4.5) or (4.7) in which the replacement $\Lambda_1 \to \Lambda_1^{-1}$, $\Lambda_2 \to \Lambda_2^{-1}$ has been made, and retain the part of degree 0 in Λ_1 and in Λ_2. We do not reproduce the result here (it is very complicated), but merely note that, apart from subgroup Casimirs, it has six denominator factors, implying six functionally independent subgroup labelling operators.

4.4. *SU(4); $(\mu_1, 0, 0)$ and $(0, 0, \mu_3)$*

The generating function for tensors in the non-degenerate enveloping algebra is given by I, equation (4.2).

The number of independent λ tensors in the degenerate enveloping algebra acting on $(\mu_1, 0, 0)$ or $(0, 0, \mu_3)$ representations is equal to the number of SU(3) × U(1) scalars in the representation λ. From the known branching rules for SU(4) ⊃ SU(3) × U(1) we get the generating function

$$[(1-\Lambda_1 PZ^{1/4})(1-\Lambda_1 Z^{-3/4})(1-\Lambda_2 QZ^{1/2}) \times(1-\Lambda_2 PZ^{-1/2})(1-\Lambda_3 Z^{3/4})(1-\Lambda_3 QZ^{-1/4})]^{-1} \quad (4.9)$$

for SU(3) × U(1) tensors in SU(4) representations. P, Q carry the representation labels of SU(3) and Z that of U(1). The generating function for SU(3) × U(1) scalars in SU(4) representations is obtained from (4.9) by setting $P=Q=0$ and retaining the part of degree 0 in Z. The result is $(1-\Lambda_1\Lambda_3)^{-1}$. Since (101) is the adjoint representation of SU(4) we conclude that $(1-U\Lambda_1\Lambda_3)^{-1}$ is the generating function for tensors in the degenerate enveloping algebra.

4.5. *SU(4); $(0, \mu_2, 0)$*

The number of independent λ tensors in the enveloping algebra acting on degenerate representations $(0, \mu_2, 0)$ is equal to the number of SU(2) × SU(2) × U(1) scalars in the representation λ. An integrity basis for SU(4) ⊃ SU(2) × SU(2) × U(1) branching rules is given by Sharp (1972). The corresponding generating function is

$$[(1-\Lambda_1 SZ)(1-\Lambda_1 TZ^{-1})(1-\Lambda_2 Z^2)(1-\Lambda_2 Z^{-2})(1-\Lambda_3 TZ)(1-\Lambda_3 SZ^{-1})]^{-1} \times[(1-\Lambda_1\Lambda_3)^{-1}+\Lambda_2 ST(1-\Lambda_2 ST)^{-1}], \quad (4.10)$$

where Λ_1, Λ_2, Λ_3 carry the SU(4) representation labels, S, T carry the SU(2) × SU(2) labels and Z carries the U(1) lavel as an exponent. The SU(2) × SU(2) × U(1) scalar part of (4.10) is found by setting $S=T=0$ and keeping the part of degree zero in Z. The result is

$$[(1-\Lambda_2^2)(1-\Lambda_1\Lambda_3)]^{-1}. \quad (4.11)$$

Comparing this with the non-degenerate generating function we conclude that, apart from scalars, the degenerate generating function is

$$[(1-U^a\Lambda_2^2)(1-U\Lambda_1\Lambda_3)]^{-1} \quad (4.12)$$

where $a=2$ or 4.

Now the generating function (4.10) for $SU(4)\supset SU(2)\times SU(2)\times U(1)$ with $\Lambda_1=\Lambda_2=0$ reduces to $[(1-\Lambda_2Z^2)(1-\Lambda_2Z^{-2})(1-\Lambda_2ST)]^{-1}$, which shows that the generating function for $SU(2)\times SU(2)\times U(1)$ scalars in the enveloping algebra is

$$[(1-U)(1-U^2)]^{-1} \tag{4.13}$$

(one SU(2) Casimir and the generator whose eigenvalue is the U(1) label). According to (4.11) we obtain a generating function for $SU(2)\times SU(2)\times U(1)$ scalars in the degenerate enveloping algebra by retaining the part of (4.12) even in Λ_2 and of equal degree in Λ_1, Λ_3 and setting $\Lambda_1=\Lambda_2=\Lambda_3=1$. The result, $[(1-U^a)(1-U)]^{-1}$, on comparison with (4.13), shows that the desired generating function for tensors in the degenerate enveloping algebra is

$$[(1-U^2\Lambda_2^2)(1-U\Lambda_1\Lambda_3)]^{-1}. \tag{4.14}$$

4.6. $SU(4)$, $(\mu_1, \mu_2, 0)$, $(\mu_1, 0, \mu_3)$, $(0, \mu_2, \mu_3)$

The number of independent λ tensors in the enveloping algebra acting on representations for which just one representation vanishes is equal to the number of $SU(2)\times U(1)\times U(1)$ scalars in the representation λ. The corresponding generating function is obtained from (4.9) by (see the discussion in § 3.1) retaining the part of equal degree in P and Q, with P and Q then set equal to unity, and of degree 0 in Z. The result,

$$[(1-\Lambda_1\Lambda_3)^2(1-\Lambda_2^2)]^{-1}[(1-\Lambda_1^2\Lambda_2)^{-1}+\Lambda_2\Lambda_3^2(1-\Lambda_2\Lambda_3^2)^{-1}], \tag{4.15}$$

implies the generating function

$$[(1-U\Lambda_1\Lambda_3)(1-U^b\Lambda_1\Lambda_3)(1-U^a\Lambda_2^2)]^{-1}[(1-U^c\Lambda_1^2\Lambda_2)^{-1}+U^c\Lambda_2\Lambda_3^2(1-U^c\Lambda_2\Lambda_3^2)^{-1}] \tag{4.16}$$

with $a=2$ or 4, $b=2$ or 3, $c=3$, 4 or 5. To learn more, we consider the implication of (4.16) for $SU(2)\times SU(2)\times U(1)$ scalars in the degenerate enveloping algebra. According to the branching rules (4.10) the $SU(2)\times SU(2)\times U(1)$ representations are all independent, and there are no missing labels; hence the generating function for $SU(2)\times SU(2)\times U(1)$ scalars is

$$[(1-U)(1-U^2)^2]^{-1}. \tag{4.17}$$

According to (4.11) the $SU(2)\times SU(2)\times U(1)$ scalar part of (4.16) is obtained by retaining the part even in Λ_2 and of equal degree in Λ_1, Λ_3, with Λ_1, Λ_2, Λ_3 then set equal to unity. The result $[(1-U)(1-U^a)(1-U^b)]$ implies $a=b=2$ and tells nothing about c. No other information is obtained from scalars of other subgroups in the enveloping algebra. To determine c we constructed the highest components of the third-degree tensors $(3;0,1,2)$ and $(3;2,1,0)$ (the notation is $(u;\lambda_1,\lambda_2,\lambda_3)$ where u is the degree and (λ) is the representation); they were then applied to the lowest states of the degenerate representations with one vanishing label. The result is non-zero, from which we conclude that $c=3$ and the degenerate enveloping algebra is described by the generating function

$$[(1-U\Lambda_1\Lambda_3)(1-U^2\Lambda_1\Lambda_3)(1-U^2\Lambda_2^2)]^{-1}[(1-U^3\Lambda_1^2\Lambda_2)^{-1} + U^3\Lambda_2\Lambda_3^2(1-U^3\Lambda_2\Lambda_3^2)^{-1}]. \tag{4.18}$$

4.7. SU(4) ⊃ SU(2) × SU(2) labelling operators

The only maximal Lie subgroup of SU(4) with missing labels for degenerate representations is SU(2) × SU(2), of interest in connection with the Wigner supermultiplet model of nuclear physics, where the SU(2) subgroups refer to spin and isospin. The only degenerate representations for which there are missing labels are those with one zero label; then there is one missing label.

The generating function for SU(4) ⊃ SU(2) × SU(2) branching rules is given by Patera and Sharp (1980, equation (3.6)), with $\Lambda_4 = 0$. The subgroup scalar part is $[(1-\Lambda_1^2)(1-\Lambda_2^2)(1-\Lambda_3^2)]^{-1}$. Hence the generating function for missing label operators for the degenerate representations with one zero label is obtained from (4.18) by retaining the even part in each of Λ_1, Λ_2, Λ_3 and then setting $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$. The result is

$$(1+U^3+U^6+U^9)[(1-U^4)(1-U^6)]^{-1}. \tag{4.19}$$

We have omitted a denominator factor $(1-U^2)^{-2}$ corresponding to the subgroup Casimirs.

Quesne (1976) has given the missing label operators for the SU(4) ⊃ SU(2) × SU(2) problem. There is one of degree 3, (1, 1, 1); three of degree 4, (2, 2, 0), (2, 0, 2) and (2, 1, 1); four of degree 6, (3, 2, 1), (3, 1, 2), (4, 2, 0) and (4, 0, 2); two of degree 9, (6, 3, 0) and (6, 0, 3). The three integers which label a labelling operator are its degrees in the (1, 1), (1, 0), (0, 1) SU(2) × SU(2) tensors which comprise the SU(4) generators. The three degree-4 scalars become indistinguishable on degenerate represenatations, as do the two degree 9; of the four labelling operators of degree 6, only two are linearly independent, and the square of one can be expressed in terms of the other.

5. Concluding remarks

For any group–subgroup it is known (Peccia and Sharp 1976) that there are twice as many functionally independent missing label operators as the number of missing labels. Although the proof is valid only for general representations of the group in question, the examples of the present paper suggest it may be true also for degenerate representations. With one vanishing representation label there is one missing label for SO(5) ⊃ SU(2), SU(4) ⊃ SU(2) × SU(2), G_2 ⊃ SU(2) × SU(2) and three missing labels for G_2 ⊃ SO(3). In each of these cases we see that the generating function for missing label operators has twice as many denominator factors (i.e. there are twice as many functionally independent missing label operators) as there are missing labels. We have not been able to find a general proof of the conjecture. Incidentally, a formula for the number of internal labels of degenerate irreducible representations of compact semi-simple Lie groups has been given by Seligman and Sharp (1983).

We hope in the future to extend our results to the other rank-3 groups, SO(7) and Sp(6), and possibly to SU(5).

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